ODE Lecture Notes, Part II, for Fall Semester 2022

Revised on 2022-12-1

Remark 0.1 ***This is the second part for my 2022 fall semester ODE course.***

Remark 0.2 *This notes is based on the textbook* ***"Elementary Differential Equations & Boundary Value Problems, 10th Edition" by Boyce & DiPrima.*** *However,* ***I will not follow the book exactly****. Lecture notes will be given to you via email whenever necessary.*

Chapter 3: Second order linear equations.

Method of undetermined coefficients (this is Section 3.5 of the book). See p. 182, Table 3.5.1.

Remark 0.3 *The "method of undetermined coefficients" provides you a way to* ***"guess"*** *the form of a particular solution. Then we plug in the form into the equation to find a correct particular solution.*

In this section, we consider a nonhomogeneous second order linear equation with constant coefficients, given by

$$ay''(t) + by'(t) + cy(t) = g(t), \quad a \neq 0, \quad t \in (-\infty, \infty), \tag{1}$$

where $g(t)$ has one of the following forms

$$P_n(t)e^{\lambda t}, \qquad P_n(t)e^{\alpha t}\cos\beta t, \qquad P_n(t)e^{\alpha t}\sin\beta t. \tag{2}$$

Here $P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n,\ a_0 \neq 0$, is a **polynomial** with degree n and $\lambda,\ \alpha,\ \beta \in \mathbb{R}$ with $\beta > 0$. Note that the case $\lambda = 0$ and the case $\alpha = 0$ are allowed. In case $\lambda = 0$ and $\alpha = 0$, $P_n(t)e^{\lambda t} = P_n(t)$ is just a polynomial in t and $P_n(t)e^{\alpha t}\cos\beta t$ becomes $P_n(t)\cos\beta t$.

We know that the general solution $y(t)$ of (1) is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \qquad t \in (-\infty, \infty),$$

where $y_p(t)$ is a **particular solution** of the nonhomogeneous equation (1) and $y_1(t),\ y_2(t)$ are solutions of $ay''(t) + by'(t) + cy(t) = 0$, determined by the roots of the characteristic equation $ar^2 + br + c = 0$. Since we know how to find $y_1(t),\ y_2(t)$, it suffices to find a particular solution $y_p(t)$ of (1).

The **"method of undetermined coefficients"** says that we can try a **particular solution** of the **form** given by **Table 3.5.1 in p. 182** of the book and then plug in the form into the **nonhomogeneous equation** (1) to **determine the coefficients**. After that, one can find a particular solution $y_p(t)$.

Remark 0.4 *Explain Table 3.5.1 in p. 182 ....*

Remark 0.5 ***(Important.)*** *The function $g(t)$ in equation (1) must have the form in (2); otherwise, the* ***"method of undetermined coefficients"*** *does not work.*

Motivation of the undetermined coefficients method.

Motivation using the equation $y'(t) - \lambda y(t) = a_0 e^{\alpha t}$. One can use simple first order equation to explain the method. Consider the simple equation

$$y'(t) - \lambda y(t) = a_0 e^{\alpha t}, \quad a_0, \ \lambda, \ \alpha \text{ are constants, } a_0 \neq 0. \tag{3}$$

The **characteristic equation of the homogeneous equation** $y'(t) - \lambda y(t) = 0$ is $r - \lambda = 0$, which has root $r = \lambda$ and so the general solution of $y'(t) - \lambda y(t) = 0$ is given by $y(t) = Ce^{\lambda t}$ for arbitrary constant C. To find the general solution of (3), it suffices to find a **particular solution** $y_p(t)$.

Case 1: If $\alpha \neq \lambda$ (i.e. α is **not a root** of the characteristic equation $r - \lambda = 0$), then the function

$$\left(e^{\alpha t}\right)' - \lambda\left(e^{\alpha t}\right) = (\alpha - \lambda) e^{\alpha t}$$

is **not zero** and is still of the form $Ke^{\alpha t}$ for constant $K = \alpha - \lambda \neq 0$. This form matches with the function $a_0 e^{\alpha t}$ on the right hand side of the equation. Therefore, if we try $y_p(t)$ to have the form

$$y_p(t) = A_0 e^{\alpha t} \tag{4}$$

and **choose the coefficient** A_0 **suitably**, we can obtain a particular solution of the equation (3). To find A_0, we plug $y_p(t) = A_0 e^{\alpha t}$ into (3) and get the identity

$$(\alpha - \lambda) A_0 e^{\alpha t} = a_0 e^{\alpha t}, \quad \alpha - \lambda \neq 0, \quad a_0 \neq 0. \tag{5}$$

Hence, if we choose $A = \frac{a_0}{\alpha - \lambda}$ (denominator is not zero), we can obtain a particular solution $y_p(t) = \frac{a_0}{\alpha - \lambda} e^{\alpha t}$ of (3). Thus the general solution of (3) is

$$y(t) = Ce^{\lambda t} + \frac{a_0}{\alpha - \lambda} e^{\alpha t}, \quad t \in (-\infty, \infty), \quad C \text{ is arbitrary const..} \tag{6}$$

Case 2: If $\alpha = \lambda$ (i.e. α is **a root** of the characteristic equation $r - \lambda = 0$), then identity (5) will becomes $0 = a_0 e^{\alpha t}$, which is impossible and it suggests that we cannot try $y_p(t)$ to have the form $y_p(t) = A_0 e^{\alpha t}$. instead, if we try

$$y_p(t) = t \cdot A_0 e^{\alpha t}, \tag{7}$$

and plug it into (3), we get the identity

$$A_0 e^{\alpha t} + \alpha A_0 t e^{\alpha t} - \lambda A_0 t e^{\alpha t} = a_0 e^{\alpha t} \quad (\text{note that } \alpha = \lambda).$$

Hence if we choose $A_0 = a_0$, the function $y_p(t) = t \cdot a_0 e^{\alpha t}$ will be a particular solution of (3) and from this we can obtain general solution of (3).

Motivation using the equation $y'(t) - \lambda y(t) = (a_0 + b_0 t) e^{\alpha t}$. One step further, now we look at the equation

$$y'(t) - \lambda y(t) = (a_0 + b_0 t) e^{\alpha t}, \quad a_0, \ b_0, \ \lambda, \ \alpha \text{ are constants, } a_0 \neq 0, \ b_0 \neq 0. \tag{8}$$

Case 1: If $\alpha \neq \lambda$ (i.e. α is **not a root** of the characteristic equation $r - \lambda = 0$), based on the above observation, the only way you can try is

$$y_p(t) = (A_0 + B_0 t) e^{\alpha t} \quad \text{for some constants} \quad A_0, \ B_0, \tag{9}$$

and if you plug it into equation (8), you get

$$B_0 e^{\alpha t} + \alpha (A_0 + B_0 t) e^{\alpha t} - \lambda (A_0 + B_0 t) e^{\alpha t} = (a_0 + b_0 t) e^{\alpha t},$$

which is same as

$$B_0 + \alpha(A_0 + B_0 t) - \lambda(A_0 + B_0 t) = a_0 + b_0 t, \tag{10}$$

and you need to choose $A_0,\ B_0$ satisfying

$$\begin{cases} B_0 + (\alpha - \lambda)\, A_0 = a_0 \\ (\alpha - \lambda)\, B_0 = b_0, \quad \alpha - \lambda \neq 0, \end{cases}$$

and conclude that if we choose

$$A_0 = \frac{a_0}{\alpha - \lambda} - \frac{b_0}{(\alpha - \lambda)^2}, \qquad B_0 = \frac{b_0}{\alpha - \lambda}, \quad \alpha \neq \lambda, \tag{11}$$

then $y_p(t)$ in (9) will be a **particular solution** of the ODE (8).

Case 2: If $\alpha = \lambda$ (i.e. α is **a root** of the characteristic equation $r - \lambda = 0$), then the identity (10) becomes $B_0 = a_0 + b_0 t$, which is **impossible to hold**. Therefore you need to modify your choice of $y_p(t)$ in (9). A natural next choice is (increase the order of the coefficient polynomial) to try:

$$y_p(t) = (A_0 + B_0 t + C_0 t^2)e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0,\ C_0.$$

However, note that $A_0 e^{\alpha t}$ is already a solution of the homogeneous equation $y'(t) - \lambda y(t) = 0$, there is **no need** to include it. Hence we choose

$$y_p(t) = (B_0 t + C_0 t^2)e^{\alpha t} = t\,(B_0 + C_0 t)\, e^{\alpha t}$$

and for consistency of notations, we write it as

$$y_p(t) = t \cdot (A_0 + B_0 t)\, e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0. \tag{12}$$

If you plug the above $y_p(t)$ into (8), you get

$$(A_0 + B_0 t)\, e^{\alpha t} + t B_0 e^{\alpha t} = (a_0 + b_0 t)e^{\alpha t}$$

and conclude

$$A_0 = a_0, \quad B_0 = \frac{b_0}{2}.$$

Thus when $\alpha = \lambda$, the function

$$y_p(t) = t \cdot \left(a_0 + \frac{b_0}{2} t\right) e^{\alpha t}, \quad t \in (-\infty, \infty)$$

will be a **particular solution** of the equation (8).

We can summarize the above method in the following:

Lemma 0.6 ***(Motivation of the undetermined coefficients method via first-order equation.)*** *Consider the first order nonhomogeneous linear equation*

$$y'(t) - \lambda y(t) = a_0 e^{\alpha t}, \quad a_0,\ \lambda,\ \alpha \text{ are constants, } a_0 \neq 0. \tag{13}$$

If $\alpha \neq \lambda$ (i.e. α is ***not a root*** *of the characteristic equation $r - \lambda = 0$), then a particular solution $y_p(t)$ of (13) has the form*

$$y_p(t) = A_0 e^{\alpha t} \quad \text{for some constant} \quad A_0. \tag{14}$$

*If $\alpha = \lambda$ (i.e. α is **a root** of the characteristic equation $r - \lambda = 0$), then a particular solution $y_p(t)$ of (13) has the form*

$$y_p(t) = t \cdot A_0 e^{\alpha t} \quad \text{for some constant} \quad A_0. \tag{15}$$

Similarly, if we consider the first order nonhomogeneous linear equation

$$y'(t) - \lambda y(t) = (a_0 + b_0 t)e^{\alpha t}, \quad a_0,\ b_0,\ \lambda,\ \alpha \text{ are constants, } a_0 \neq 0,\ b_0 \neq 0. \tag{16}$$

*If $\alpha \neq \lambda$ (i.e. α is **not a root** of the characteristic equation $r - \lambda = 0$), then a particular solution $y_p(t)$ of (16) has the form*

$$y_p(t) = (A_0 + B_0 t)e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0, \tag{17}$$

*If $\alpha = \lambda$ (i.e. α is **a root** of the characteristic equation $r - \lambda = 0$), then a particular solution $y_p(t)$ of (16) has the form*

$$y_p(t) = t \cdot (A_0 + B_0 t)e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0. \tag{18}$$

From Lemma 0.6, you can understand the undetermined coefficients method in Table 3.5.1 in p. 182 of the book.

Remark 0.7 *State the rule in Table 3.5.1 in p. 182 of the textbook here.*

P. 183, Case 2. (Read this section by yourself.)

Remark 0.8 *This section gives a **detailed proof** on Case 2 in p. 183 of the textbook, showing that the method **does work** !! If you are interested, you can read it by yourself.*

This is to verify that the **method of undetermined coefficients** can be used to solve a nonhomogeneous second order linear ODE (with constant coefficients) of the form

$$ay''(t) + by'(t) + cy(t) = P_n(t)\, e^{\lambda t}, \quad a \neq 0, \tag{19}$$

where

$$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$$

is a polynomial with degree n.

Remark 0.9 *Of course, one can also use reduction method to solve (19), but the **method of undetermined coefficients** will be easier for $g(t)$ of the form $P_n(t)\, e^{\lambda t}$.*

We let $y_p(t) = u(t)\, e^{\lambda t}$ be the particular solution to be found (there is no other better try than this), where $u(t)$ is to be determined. Plug $y_p(t) = u(t)\, e^{\lambda t}$ into (19) to get

$$a\left[u''(t)\, e^{\lambda t} + 2u'(t)\, \lambda e^{\lambda t} + u(t)\, \lambda^2 e^{\lambda t}\right] + b\left[u'(t)\, e^{\lambda t} + u(t)\, \lambda e^{\lambda t}\right] + cu(t)\, e^{\lambda t} = P_n(t)\, e^{\lambda t}.$$

We can cancel $e^{\lambda t}$ and the equation becomes

$$\underbrace{au''(t) + (2a\lambda + b)\, u'(t) + \left(a\lambda^2 + b\lambda + c\right) u(t)} \\ = P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n. \tag{20}$$

Assume first that λ is **not a root of the characteristic equation** $ar^2 + br + c = 0$. Hence $a\lambda^2 + b\lambda + c \neq 0$. One can try

$$u(t) = A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_n. \tag{21}$$

Note that

$$\begin{cases} u'(t) = nA_0t^{n-1} + (n-1)A_1t^{n-2} + \cdots + 2A_{n-2}t + A_{n-1} \\ u''(t) = n(n-1)A_0t^{n-2} + (n-1)(n-2)A_1t^{n-3} + \cdots + 2A_{n-2}. \end{cases}$$

If we plug (21) into (20) and compare coefficients, we can get the following system of equations (note that $P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n$):

$$\begin{cases} (a\lambda^2 + b\lambda + c)A_0 = a_0 \text{ (coefficients of } t^n\text{)}, \quad \text{where} \quad a\lambda^2 + b\lambda + c \neq 0 \\ (2a\lambda + b)nA_0 + (a\lambda^2 + b\lambda + c)A_1 = a_1 \text{ (coefficients of } t^{n-1}\text{)} \\ an(n-1)A_0 + (2a\lambda + b)(n-1)A_1 + (a\lambda^2 + b\lambda + c)A_2 = a_2 \text{ (coefficients of } t^{n-2}\text{)} \\ \cdots \\ a2A_{n-2} + (2a\lambda + b)A_{n-1} + (a\lambda^2 + b\lambda + c)A_n = a_n \text{ (coefficients of } t^0\text{)}. \end{cases} \tag{22}$$

Then one can solve all A_0, ..., A_n and obtain $u(t)$, and conclude that $y(t) = u(t)e^{\lambda t}$ is a solution of the nonhomogeneous equation (19).

If λ is **a root with multiplicity** $s = 1$, then $a\lambda^2 + b\lambda + c = 0$ and $2a\lambda + b \neq 0$. The above trial solution (21) **does not work out**. Instead we try

$$u(t) = t\left(A_0t^n + A_1t^{n-1} + \cdots + A_{n-1}t + A_n\right) = A_0t^{n+1} + A_1t^n + \cdots + A_{n-1}t^2 + A_nt$$

Then (20) becomes

$$\underbrace{au''(t) + (2a\lambda + b)u'(t)} = P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n \tag{23}$$

and (22) becomes

$$\begin{cases} (2a\lambda + b)(n+1)A_0 = a_0 \text{ (coefficients of } t^n\text{)}, \quad \text{where} \quad 2a\lambda + b \neq 0 \\ an(n+1)A_0 + (2a\lambda + b)nA_1 = a_1 \text{ (coefficients of } t^{n-1}\text{)} \\ \cdots \\ a2A_{n-2} + (2a\lambda + b)A_n = a_n \text{ (coefficients of } t^0\text{)}. \end{cases} \tag{24}$$

In this case we can solve all A_0, ..., A_n and conclude that $y(t) = u(t)e^{\lambda t}$ is a solution of (19).

Finally if λ is **a root with multiplicity** $s = 2$, then $a\lambda^2 + b\lambda + c = 0$ and $2a\lambda + b = 0$, but $a \neq 0$. Then we try

$$u(t) = t^2\left(A_0t^n + A_1t^{n-1} + \cdots + A_{n-1}t + A_n\right) = A_0t^{n+2} + A_1t^{n+1} + \cdots + A_nt^2.$$

Now (20) becomes

$$\underbrace{au''(t)} = P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n \tag{25}$$

and (22) becomes

$$\begin{cases} a(n+2)(n+1)A_0 = a_0 \text{ (coefficients of } t^n\text{)}, \quad \text{where} \quad a \neq 0 \\ an(n+1)A_1 = a_1 \text{ (coefficients of } t^{n-1}\text{)} \\ \cdots \\ a2A_n = a_n \text{ (coefficients of } t^0\text{)}. \end{cases} \tag{26}$$

Again, we can solve all A_0, ..., A_n and obtain a particular solution of (19).

In conclusion, the method works for the case $g(t) = P_n(t)e^{\lambda t}$, $\lambda \in \mathbb{R}$. The verification is done.□

Example 0.10 $y'' + 3y = 4e^{-5t}, \quad y_p(t) = \frac{1}{7}e^{-5t}.$

Example 0.11 $y'' - 3y' - 4y = 2e^{-t}, \quad y_p(t) = -\frac{2}{5}te^{-t}.$

Example 0.12 $y'' + 2y = \sin 3t, \quad y_p(t) = -\frac{1}{7}\sin 3t$ *(since there is no first order term y', the solution $y_p(t)$ is also of the form $\sin 3t$).*

Example 0.13 $y'' + 9y = \sin 3t, \quad y_p(t) = -\frac{1}{6}t\cos 3t.$

Example 0.14 $y'' - 3y = t^2, \quad y_p(t) = -\frac{1}{3}t^2 - \frac{2}{9}.$

Example 0.15 $y'' - 3y' = t + t^2, \quad y_p(t) = t\left(-\frac{1}{9}t^2 - \frac{5}{18}t - \frac{5}{27}\right).$

Example 0.16 *Do Example 3 in p. 179.*

Example 0.17 *Find general solution of the equation*

$$y'' + 2y' + y = te^{-t}.$$

Solution:

By the rule for $y_p(t)$, it has the form

$$y_p(t) = t^s(At + B)e^{-t} = \left(At^3 + Bt^2\right)e^{-t}, \quad \text{where } s = 2.$$

Plugging it into equation to get

$$\left\{\begin{array}{l} \left[(6At + 2B)e^{-t} - 2\left(3At^2 + 2Bt\right)e^{-t} + \left(At^3 + Bt^2\right)e^{-t}\right] \\ +2\left[\left(3At^2 + 2Bt\right)e^{-t} - \left(At^3 + Bt^2\right)e^{-t}\right] + \left(At^3 + Bt^2\right)e^{-t} \end{array}\right. = te^{-t}.$$

Hence, after simplification, we need to solve $6At + 2B = t$, which gives

$$A = \frac{1}{6}, \quad B = 0.$$

Thus $y_p(t) = \frac{1}{6}t^3e^{-t}$ is a particular solution of the equation. The general solution is

$$y(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{6}t^3e^{-t}, \quad t \in (-\infty, \infty).$$

□

Remark 0.18 *If an equation has the form*

$$ay'' + by' + cy = f(t) + g(t), \tag{27}$$

where $f(t)$ and $g(t)$ both have the form in the above case 1 or case 2 (say $f(t) = t^2e^{5t}$ and $g(t) = \left(t^3 + 2t^2 - 6t - 5\right)e^{-t}\cos 7t$), then use the undetermined coefficients to find $y_p(t)$ for the equation

$$ay'' + by' + cy = f(t)$$

and then use the same method to find $\tilde{y}_p(t)$ for the equation

$$ay'' + by' + cy = g(t).$$

Then the general solution of (27) is given by

$$x(t) = y_p(t) + \tilde{y}_p(t) + c_1y_1(t) + c_2y_2(t),$$

where $c_1x_1(t) + c_2x_2(t)$ is the general solution of the corresponding homogeneous equation.

Example 0.19 *Find the correct form of a particular solution of the equation*

$$y'' - 4y' + 4y = 3t^2 e^{2t} + 2t \sin t - 8e^t \cos 2t.$$

Solution:

The correct form is

$$y_p(t) = \underbrace{t^2 \left(At^2 + Bt + C\right) e^{2t}} + \underbrace{(Dt + E) \cos t + (Ft + G) \sin t} + \underbrace{Ke^t \cos 2t + Le^t \sin 2t},$$

where A, ..., L are constant coefficients to be determined. □

Example 0.20 *(**This is Exercise 30 in p. 185 with one extra term.**) Find the general solution of the equation*

$$y'' + \lambda^2 y = \sum_{m=1}^{N} (a_m \sin m\pi t + b_m \cos m\pi t), \quad t \in (-\infty, \infty), \tag{28}$$

where $\lambda > 0$ *and* $\lambda \neq m\pi$ *for* $m = 1, 2, ..., N$.

Solution:

The two roots of the characteristic polynomial $r^2 + \lambda^2 = 0$ are $r = \pm\lambda i$, where $\lambda \neq m\pi$ for any $m = 1, ..., N$. Hence **for each** fixed $m = 1, ..., N$, we try a particular solution $y_m(t)$ of the form

$$y_m(t) = A_m \sin m\pi t + B_m \cos m\pi t, \tag{29}$$

which is for the equation

$$y'' + \lambda^2 y = a_m \sin m\pi t + b_m \cos m\pi t. \tag{30}$$

We plug the above $y_m(t)$ into equation (30) to get

$$\left(\lambda^2 - m^2\pi^2\right) A_m \sin m\pi t + \left(\lambda^2 - m^2\pi^2\right) B_m \cos m\pi t = a_m \sin m\pi t + b_m \cos m\pi t$$

and obtain

$$A_m = \frac{a_m}{\lambda^2 - m^2\pi^2}, \qquad B_m = \frac{b_m}{\lambda^2 - m^2\pi^2}, \qquad m = 1, ..., N.$$

Hence, the general solution of the equation is given by (add all $y_m(t)$ together):

$$y(t) = c_1 \sin \lambda t + c_2 \cos \lambda t + \sum_{m=1}^{N} \left(\frac{a_m}{\lambda^2 - m^2\pi^2}\right) \sin m\pi t + \left(\frac{b_m}{\lambda^2 - m^2\pi^2}\right) \cos m\pi t.$$

The proof is done. □

Variation of parameters method (this is Section 3.6 of the book) for nonhomogeneous linear equations with variable coefficients.

Remark 0.21 *(**Be careful.**) Throughout this section, we will focus on equation (31), which has leading coefficient* 1 *for* $y''(t)$.

In this section we focus on a **nonhomogeneous linear equation with variable coefficients** (which has **leading coefficient** 1), given by

$$y'' + p(t) y' + q(t) y = g(t), \quad t \in I, \tag{31}$$

where $p(t)$, $q(t)$, $g(t)$ are given continuous function on some interval $I \subseteq \mathbb{R}$ and here the function $g(t)$ can be an **arbitrary.**

Assume we are given a **fundamental set of solutions** $y_1(t)$ and $y_2(t)$ for the corresponding homogeneous equation $y'' + p(t) y' + q(t) y = 0$ on I. To solve (31), we try a solution of the form:

$$y(t) = u_1(t) y_1(t) + u_2(t) y_2(t), \quad t \in I \tag{32}$$

and look for suitable $u_1(t)$ and $u_2(t)$. We will solve a **first-order system of ODE** for $u_1(t)$ and $u_2(t)$.

Remark 0.22 *(**Useful observation.**) One can view (32) as a generalization of the **reduction method** because if we only try $y(t) = u_1(t) y_1(t)$, it is exactly the reduction method.*

Remark 0.23 *If an equation has the form*

$$P(t) y''(t) + Q(t) y'(t) + R(t) y(t) = G(t), \quad t \in I, \quad P(t) \neq 0 \text{ on } I, \tag{33}$$

*then **you should divide the whole equation by** $P(t)$ **first and then apply the method below.***

We need to impose suitable conditions on $u_1(t)$ and $u_2(t)$ so that the above $y(t)$ is a solution of (31). We first note that

$$y'(t) = \underbrace{[u_1'(t) y_1(t) + u_2'(t) y_2(t)]} + [u_1(t) y_1'(t) + u_2(t) y_2'(t)]$$

and **impose the first condition**

$$\underbrace{u_1'(t) y_1(t) + u_2'(t) y_2(t)} = 0, \quad t \in I. \tag{34}$$

Remark 0.24 *If we impose the condition on the term $u_1(t) y_1'(t) + u_2(t) y_2'(t)$, then in $y''(t)$ we will encounter $u_1''(t)$ and $u_2''(t)$. With this, the method will not work at all.*

Then, under the assumption of (34), $y'(t)$ becomes

$$y'(t) = u_1(t) y_1'(t) + u_2(t) y_2'(t), \quad t \in I$$

and so

$$y''(t) = \underbrace{[u_1'(t) y_1'(t) + u_2'(t) y_2'(t)]} + [u_1(t) y_1''(t) + u_2(t) y_2''(t)], \quad t \in I.$$

Then we **impose the second condition** as

$$\underbrace{u_1'(t) y_1'(t) + u_2'(t) y_2'(t)} = g(t). \tag{35}$$

Under the assumption of (34) and (35), we conclude

$$\begin{cases} y'(t) = u_1(t) y_1'(t) + u_2(t) y_2'(t) \\ y''(t) = g(t) + u_1(t) y_1''(t) + u_2(t) y_2''(t) \end{cases}$$

and so

$$\begin{aligned}
& y''(t) + p(t) y'(t) + q(t) y(t) \\
&= [g(t) + u_1(t) y_1''(t) + u_2(t) y_2''(t)] + p(t) [u_1(t) y_1'(t) + u_2(t) y_2'(t)] + q(t) [u_1(t) y_1(t) + u_2(t) y_2(t)] \\
&= g(t) + u_1(t) \left[\underbrace{y_1''(t) + p(t) y_1'(t) + q(t) y_1(t)}\right] + u_2(t) \left[\underbrace{y_2''(t) + p(t) y_2'(t) + q(t) y_2(t)}\right] \\
&= g(t) + u_1(t) \cdot 0 + u_2(t) \cdot 0 = g(t), \quad t \in I,
\end{aligned}$$

which says that $y(t)$ is indeed a solution of the equation (31).

It remains to claim that (34) and (35) can be satisfied. For this purpose, we need to solve the following **first-order system of ODE** for $u_1(t)$ and $u_2(t)$:

$$\begin{cases} u_1'(t)\,y_1(t) + u_2'(t)\,y_2(t) = 0 \\ u_1'(t)\,y_1'(t) + u_2'(t)\,y_2'(t) = g(t) \end{cases} \tag{36}$$

and get

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = -\frac{y_2(t)\,g(t)}{W(t)}, \qquad u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & g(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{y_1(t)\,g(t)}{W(t)},$$

where $W(t) = W(y_1, y_2)(t)$ is the **Wronskian** of $y_1(t)$ and $y_2(t)$ on I.

The above gives

$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(t)}dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)\,g(t)}{W(t)}dt + c_2, \tag{37}$$

and the general solution of (31) is given by

$$\begin{aligned} y(t) &= \left(-\int \frac{y_2(t)\,g(t)}{W(t)}dt + c_1\right) y_1(t) + \left(\int \frac{y_1(t)\,g(t)}{W(t)}dt + c_2\right) y_2(t) \\ &= c_1 y_1(t) + c_2 y_2(t) + y_p(t), \end{aligned}$$

where

$$y_p(t) = -\left(\int \frac{y_2(t)\,g(t)}{W(t)}dt\right) y_1(t) + \left(\int \frac{y_1(t)\,g(t)}{W(t)}dt\right) y_2(t) \tag{38}$$

is a **particular solution** of (31). The above method is called "**variation of parameters**" method. It is a powerful method.

Remark 0.25 ***(Important.)*** *If the equation (31) has initial conditions $y(t_0) = y_0,\ y'(t_0) = z_0,\ t_0 \in I$, then there are two ways to find the unique solution $y(t)$. (1). If you know $y_p(t)$* ***explicitly****, use the formula $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$ to find $c_1,\ c_2$. (2). If you* ***do not know*** *$y_p(t)$ explicitly, you can use definite integrals to write the general solution $y(t)$ as*

$$y(t) = \begin{cases} c_1 y_1(t) + c_2 y_2(t) \\ + \left[-\left(\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(s)}ds\right) y_1(t) + \left(\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(s)}ds\right) y_2(t) \right], \quad t \in I \end{cases} \tag{39}$$

and then require $c_1,\ c_2$ to satisfy the following

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = z_0. \end{cases}$$

This is due to the fact that the ***particular solution***

$$y_p(t) = -\left(\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(s)}ds\right) y_1(t) + \left(\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(s)}ds\right) y_2(t), \quad t \in I \tag{40}$$

satisfies

$$y(t_0) = y'(t_0) = 0. \tag{41}$$

To see this, we clearly have $y_p(t_0) = 0$. As for $y_p'(t_0) = 0$, we note that

$$y_p'(t_0) = -\left(\frac{y_2(t_0)\,g(t_0)}{W(t_0)}\right) y_1(t_0) + 0 \cdot y_1'(t_0) + \left(\frac{y_1(t_0)\,g(t_0)}{W(t_0)}\right) y_2(t_0) + 0 \cdot y_2'(t_0) = 0. \tag{42}$$

Example 0.26 ***(This is Example 1 in p. 186.)*** *Solve the equation*

$$y''(t) + 4y(t) = 3\csc t, \quad 0 < t < \pi. \tag{43}$$

Solution:

Read the solution in the textbook by yourself. □

Example 0.27 ***(This is Exercise 5 in p. 190.)*** *Solve the equation*

$$y''(t) + y(t) = 2\tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}. \tag{44}$$

Remark 0.28 *Note that one cannot use the undetermined coefficients method to solve (44).*

Solution:

Since we know two independent solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of $y''(t) + y(t) = 0$, we can use variation of parameters method. We first compute

$$W(t) = W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

and by (38) we know that the general solution of (44) is given by

$$y(t) = c_1 \cos t + c_2 \sin t + \left(-\int \frac{y_2(t) \cdot 2\tan t}{W(t)} dt\right) y_1(t) + \left(\int \frac{y_1(t) \cdot 2\tan t}{W(t)} dt\right) y_2(t),$$

where

$$-\int \frac{y_2(t) \cdot 2\tan t}{W(t)} dt = -\int (\sin t)(2\tan t)\, dt = -2\int \frac{(1-\cos^2 t)}{\cos} dt = -2\int (\sec t - \cos t)\, dt$$

and

$$\int \frac{y_1(t) \cdot 2\tan t}{W(t)} dt = \int (\cos t)(2\tan t)\, dt = 2\int \sin t dt.$$

We conclude

$$\begin{aligned} y(t) &= c_1 \cos t + c_2 \sin t + \left(-2\int (\sec t - \cos t)\, dt\right) \cos t + \left(2\int \sin t dt\right) \sin t \\ &= c_1 \cos t + c_2 \sin t + \left(-2\int \sec t dt + 2\sin t\right) \cos t + (-2\cos t) \sin t \\ &= c_1 \cos t + c_2 \sin t + (-2\log|\sec t + \tan t|) \cos t. \end{aligned} \tag{45}$$

□

Remark 0.29 ***(Compare with the reduction method.)*** ***(Read this remark by yourself.)*** *If we use reduction method, we can let $y(t) = v(t)\sin t$ ($\sin t$ is a solution of $y''(t) + y(t) = 0$) and get*

$$v''(t)\sin t + 2v'(t)\cos t - v(t)\sin t + v(t)\sin t = 2\tan t,$$

which gives (let $w(t) = v'(t)$)

$$w'(t) + 2\frac{\cos t}{\sin t} \cdot w(t) = \frac{2}{\cos t}, \quad 0 < t < \frac{\pi}{2}$$

and then

$$w(t) = v'(t) = e^{-\int \frac{2\cos}{\sin t}dt}\left(\int e^{\int \frac{2\cos}{\sin t}dt}\frac{2}{\cos t}dt + C\right) = \frac{1}{\sin^2 t}\left(2\int \sin t \tan t dt + C\right)$$
$$= \frac{1}{\sin^2 t}\left(2\int (\sec t - \cos t)\, dt + C\right) = \frac{C}{\sin^2 t} + \frac{1}{\sin^2 t}\left(2\log|\sec t + \tan t| - 2\sin t\right).$$

Finally, we have

$$v(t) = \int \frac{C}{\sin^2 t}dt + \int \frac{1}{\sin^2 t}\left(2\log|\sec t + \tan t| - 2\sin t\right)dt + K$$
$$= -C\cot t + K + \underbrace{\int \frac{1}{\sin^2 t}\left(2\log|\sec t + \tan t|\right)dt} - 2\int \frac{1}{\sin t}dt,$$

where, by the integration by parts, we find

$$\underbrace{\int \frac{1}{\sin^2 t}\left(2\log|\sec t + \tan t|\right)dt} = -\int \left(2\log|\sec t + \tan t|\right)d(\cot t)$$
$$= \underbrace{-\left(2\log|\sec t + \tan t|\right)(\cot t) + 2\int \frac{1}{\sin t}dt}$$

and conclude

$$y(t) = v(t)\sin t = y(t) = \left[-C\cot t + K - \left(2\log|\sec t + \tan t|\right)(\cot t)\right]\sin t$$
$$= c_1\cos t + c_2\sin t - \left(2\log|\sec t + \tan t|\right)\cos t. \tag{46}$$

We see that (46) is the same as (45). This method clearly involves more computations. ***This is because we only make use of one solution*** $\sin t$.

Example 0.30 ***(This is Exercise 10 in p. 190.)*** *Solve the equation*

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{1+t^2}, \quad t \in (-\infty, \infty).$$

Solution:

Since we know two independent solutions $y_1(t) = e^t$ and $y_2(t) = te^t$ of $y''(t) - 2y'(t) + y(t) = 0$, we can use variation of parameters method. We first compute

$$W(t) = W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$

and so

$$y(t) = \left(-\int \frac{te^t \cdot \frac{e^t}{1+t^2}}{e^{2t}}dt + c_1\right)e^t + \left(\int \frac{e^t \cdot \frac{e^t}{1+t^2}}{e^{2t}}dt + c_2\right)te^t$$
$$= \left(-\int \frac{t}{1+t^2}dt + c_1\right)e^t + \left(\int \frac{1}{1+t^2}dt + c_2\right)te^t$$
$$= \left(-\frac{1}{2}\log\left(1+t^2\right) + c_1\right)e^t + \left(\tan^{-1}t + c_2\right)te^t,$$

which is the general solution. □

Example 0.31 *Find the general solution of the equation*

$$ty''(t) - (1+t)\,y'(t) + y(t) = t^2 e^{2t}, \quad t \in (0,\infty), \tag{47}$$

given that $y_1(t) = 1+t$ and $y_2(t) = e^t$ is a pair of ***fundamental solutions*** *for the corresponding homogeneous equation.*

<u>**Solution**</u>:

To apply the variation of parameters method, **we need to rewrite the equation to have leading coefficient of $y''(t)$ equal to** 1. We have

$$y''(t) - \left(\frac{1+t}{t}\right) y'(t) + \frac{1}{t} y(t) = te^{2t}, \quad t \in (0,\infty),$$

and obtain $g(t) = te^{2t}$. By the variation of parameters method, we have

$$y_p(t) = \left(-\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}dt\right) y_1(t) + \left(\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}dt\right) y_2(t), \quad t \in (0,\infty),$$

where

$$W(y_1,y_2)(t) = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = te^t.$$

Hence

$$\begin{aligned} y_p(t) &= \left(-\int \frac{e^t \cdot te^{2t}}{te^t}dt\right)(1+t) + \left(\int \frac{(1+t)\cdot te^{2t}}{te^t}dt\right)e^t \\ &= \left(-\int e^{2t}dt\right)(1+t) + \left(\int (1+t)\,e^t dt\right)e^t \\ &= \left(-\frac{1}{2}e^{2t}\right)(1+t) + \left(te^t\right)e^t = \frac{1}{2}(t-1)\,e^{2t}. \end{aligned}$$

and conclude the general solution for equation (47):

$$y(t) = C_1(1+t) + C_2 e^t + \frac{1}{2}(t-1)\,e^{2t}, \quad t \in (0,\infty).$$

□

An interesting equation from "mechanics of vibrations".

Consider the equation

$$y''(t) + y(t) = g(t), \quad g(t) \text{ is continuous on } I$$

with initial condition

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \quad t_0 \in I.$$

This equation appears frequently in **mechanics of vibrations**. If we choose $y_1(t) = \cos t$, $y_2(t) = \sin t$, then we have $W(t) = W(y_1,y_2)(t) = 1$ and by Remark 0.25 the **particular solution** $y_p(t)$ (with $y_p(t_0) = y_p'(t_0) = 0$) in (40) is given by

$$\begin{aligned} y_p(t) &= -\left(\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(s)}ds\right) y_1(t) + \left(\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(s)}ds\right) y_2(t), \quad W(s) \equiv 1 \\ &= -(\cos t)\int_{t_0}^{t} g(s)\sin s\,ds + (\sin t)\int_{t_0}^{t} g(s)\cos s\,ds \\ &= \int_{t_0}^{t} g(s)(\sin t\cos s - \cos t\sin s)\,ds = \int_{t_0}^{t} g(s)\sin(t-s)\,ds, \quad t \in I. \end{aligned} \tag{48}$$

The general solution of the homogeneous equation $y''(t)+y(t)=0$ satisfying the initial condition $y(t_0)=y_0,\ y'(t_0)=z_0$, is given by

$$c_1 y_1(t)+c_2 y_2(t)=c_1\cos t+c_2\sin t,\quad t\in I$$

and we need to find $c_1,\ c_2$ satisfying

$$\begin{cases} c_1\cos t_0+c_2\sin t_0=y_0 \\ -c_1\sin t_0+c_2\cos t_0=z_0, \end{cases}$$

which gives

$$c_1=y_0\cos t_0-z_0\sin t_0,\quad c_2=y_0\sin t_0+z_0\cos t_0,$$

and then

$$\begin{aligned} & c_1 y_1(t)+c_2 y_2(t) \\ & =(y_0\cos t_0-z_0\sin t_0)\cos t+(y_0\sin t_0+z_0\cos t_0)\sin t \\ & =y_0\cos(t-t_0)+z_0\sin(t-t_0). \end{aligned} \tag{49}$$

Therefore the solution satisfying the initial condition is given by the nice **solution formula**:

$$y(t)=\underbrace{y_0\cos(t-t_0)+z_0\sin(t-t_0)}+\underbrace{\int_{t_0}^{t}g(s)\sin(t-s)\,ds},\quad t\in I. \tag{50}$$

Now assume that $I=(-\infty,\infty)$ and $g(t)$ is a 2π**-periodic function** defined on $(-\infty,\infty)$ ($g(t)$ usually comes from the **external force** acting on the mechanical system, say string vibration). **The particular solution** $y_p(t)$ **in (48) may not be** 2π**-periodic in general** (but the homogeneous part $y_0\cos(t-t_0)+z_0\sin(t-t_0)$ is clearly 2π-periodic). Note that we have

$$\begin{aligned} & y_p(t+2\pi)-y_p(t) \\ & =\int_{t_0}^{t+2\pi}g(s)\sin(t+2\pi-s)\,ds-\int_{t_0}^{t}g(s)\sin(t-s)\,ds=\int_{t}^{t+2\pi}g(s)\sin(t-s)\,ds \\ & =-(\cos t)\underbrace{\int_{t}^{t+2\pi}g(s)\sin s ds}+(\sin t)\underbrace{\int_{t}^{t+2\pi}g(s)\cos s ds} \\ & =-(\cos t)\int_{0}^{2\pi}g(s)\sin s ds+(\sin t)\int_{0}^{2\pi}g(s)\cos s ds \\ & =\left\langle(-\cos t,\sin t),\left(\int_{0}^{2\pi}g(s)\sin s ds,\int_{0}^{2\pi}g(s)\cos s ds\right)\right\rangle \end{aligned}$$

and so we have $y_p(t+2\pi)=y_p(t)$ for all $t\in(-\infty,\infty)$ **if and only if** the 2π-periodic function $g(s)$ satisfies

$$\int_{0}^{2\pi}g(s)\sin s ds=\int_{0}^{2\pi}g(s)\cos s ds=0. \tag{51}$$

If we take $g(t)=\cos t$ ((51) is not satisfied), then one can check that $y_p(t)=\frac{1}{2}t\sin t$ is a particular solution (with $y_p(0)=y_p'(0)=0$) of the equation

$$y''(t)+y(t)=\cos t,$$

but it is **not** 2π-periodic even if $g(t)=\cos t$ is 2π-periodic. In fact, one can see that $y_p(t)$ in (48) is 2π-periodic **if and only if** $g(t)$ is 2π-periodic and satisfies (51), for example, say $g(t)=\cos 2t$.

Nonhomogeneous Euler equation.

One can combine the variation of parameters method and change of variables to solve a **nonhomogeneous Euler equation**, given by

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = f(t), \quad t \in (0, \infty), \quad \alpha, \ \beta \text{ constants}, \tag{52}$$

where $f(t)$ can be any **arbitrary** continuous function defined on $t \in (0, \infty)$. By the change of variables $x = \log t, \ x \in (-\infty, \infty)$, the above equation becomes

$$\frac{d^2 \tilde{y}}{dx^2} + (\alpha - 1) \frac{d\tilde{y}}{dx} + \beta \tilde{y}(x) = F(x), \quad x \in (-\infty, \infty), \tag{53}$$

where $\tilde{y}(x) = y(e^x)$ and $F(x) = f(e^x)$. We can know a pair of **fundamental solutions** $\{y_1(t), \ y_2(t)\}$ for $\tilde{y}''(x) + (\alpha - 1)\tilde{y}'(x) + \beta \tilde{y}(x) = 0$ and then use the **variation of parameters method** to find the general solution $\tilde{y}(x)$ of (53) and then change back to get $y(t)$. It will be the general solution of (52). On the other hand, if $F(x)$ in (53) has the forms appeared in the **undetermined coefficients method**, then you can use that method to find the general solution $\tilde{y}(x)$ of (53).

Remark 0.32 *In case the Euler equation has the form*

$$At^2 y''(t) + Bty'(t) + Cy(t) = f(t), \quad t \in (0, \infty), \quad A \neq 0, \ B, \ C \text{ are constants,}$$

then equation (53) becomes

$$A\frac{d^2 \tilde{y}}{dx^2} + (B - A) \frac{d\tilde{y}}{dx} + C\tilde{y}(x) = F(x). \tag{54}$$

Summary of solution methods for second order linear equations.

This is a summary for solving a nonhomogeneous second order linear equation.

Case 1: $ay'' + by' + cy = g(t), \ t \in I$, **where** $a, \ b, \ c$ **are constants with** $a \neq 0$ **and** $g(t)$ **is a continuous nonzero function on** I.

In this case you can easily find two independent solutions $y_1(t)$ and $y_2(t)$ of $ay'' + by' + cy = 0$.

(1). In case $g(t)$ is of the form $P_n(t)e^{\lambda t}, \ P_n(t)e^{\alpha t}\cos\beta t, \ P_n(t)e^{\alpha t}\sin\beta t$, where $P_n(t)$ is a polynomial with degree n and $\lambda, \ \alpha, \ \beta \in \mathbb{R}$ with $\beta > 0$, use the **method of undetermined coefficients** (the easiest way).

(2). In case $g(t)$ is not of the form in (1), you can use **decomposition method** (if the characteristic polynomial $ar^2 + br + c = 0$ has **two real roots**), or **reduction method**, or **variation of parameters method**. **Variation of parameters method seems to be the best one** because we know two independent solutions $y_1(t), \ y_2(t)$ of the equation $ay'' + by' + cy = 0$. Note that if you use **variation of parameters method,** you first need to **rewrite the equation as**

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = \frac{g(t)}{a}, \quad a \neq 0.$$

Case 2: $y'' + p(t)\, y' + q(t)\, y = g(t)\,,\ \ t \in I$, **where** $p(t)\,,\ q(t)\,,\ g(t)$ **are continuous functions on** I.

Remark 0.33 *(**Be careful.**) Here the coefficient of $y''(t)$ is 1.*

Here we assume that we are given one nonzero solution $y_1(t)$ of the homogeneous equation $y'' + p(t)\, y' + q(t)\, y = 0$ on I.

(1). In case $g(t) \equiv 0$ on I and we know one solution $y_1(t)$ of the **homogeneous** equation $y'' + p(t)\, y' + q(t)\, y = 0$, use **reduction method** or **Wronskian method**.

(2). In case $g(t)$ is **nonzero** on I and we know one solution $y_1(t)$ of the **homogeneous** equation $y'' + p(t)\, y' + q(t)\, y = 0$, use **reduction method.**

(3). In case $g(t)$ is **nonzero** on I and we know two independent solutions $y_1(t)\,,\ y_2(t)$ (**fundamental set** of solutions) of $y'' + p(t)\, y' + q(t)\, y = 0$ on I, use **variation of parameters method**.

Remark 0.34 *If the equation is of the form*

$$P(t)\, y''(t) + Q(t)\, y'(t) + R(t)\, y(t) = G(t)\,, \quad t \in I, \quad P(t) \neq 0 \text{ on } I, \tag{55}$$

then you should divide the whole equation by $P(t)$ first and then apply the above summary.

Chapter 4: Higher order linear equations.

Section 4.1: General theory of n-th order linear equations.

Consider the n-**th order linear equation**

$$y^{(n)}(t) + p_1(t)\, y^{(n-1)}(t) + \cdots + p_{n-1}(t)\, y'(t) + p_n(t)\, y(t) = g(t)\,, \quad t \in I \tag{56}$$

where $p_1(t)\,,\ ...,\ p_n(t)\,,\ g(t)$ are given and continuous on I. By ODE theory, any solution $y(t)$ to equation (56) is defined **on the whole interval** I.

When the **initial conditions**

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \quad \cdots, \quad y^{(n-1)}(t_0) = \gamma_0 \tag{57}$$

are given, we have the **existence and uniqueness theorem** (see Theorem 4.1.1 in p. 222 of the book). Moreover, the unique solution $y(t)$ is defined on the whole interval $t \in I$.

We now state the following:

Lemma 0.35 *(**Abel's formula for homogeneous equation.**) Let $y_1(t)\,,\ ...,\ y_n(t)$ be solutions of the homogeneous equation*

$$y^{(n)}(t) + p_1(t)\, y^{(n-1)}(t) + \cdots + p_{n-1}(t)\, y'(t) + p_n(t)\, y(t) = 0, \quad t \in I. \tag{58}$$

*Define its **Wronskian** $W(y_1,\ ...\ ,\ y_n)(t)$ as in the book (for simplicity, denote it as $W(t)$) (see p. 223 of the book). Then we have*

$$W(t) = Ce^{-\int p_1(t)dt}, \quad t \in I, \tag{59}$$

for some constant C. Therefore, either $W(t) \equiv 0$ or $W(t) \neq 0$ for all $t \in I$.

Remark 0.36 *(**Important.**) Be careful that the leading coefficient in equation (58) is 1 and also that the equation is homogeneous.*

Remark 0.37 *If $W(t) \neq 0$ on I, $\{y_1(t), ..., y_n(t)\}$ is called a* ***fundamental set of solutions*** *of equation (58) on I.*

Proof. We have

$$W(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}, \quad t \in I. \tag{60}$$

Compute

$$\frac{dW}{dt}(t) = \det(M(t)),$$

where $M(t)$ is $n \times n$ matrix whose first $n-1$ rows are **unchanged** (i.e. the same as the first $n-1$ rows of (60)) and whose n-th row is $\left(y_1^{(n)}(t), ..., y_n^{(n)}(t)\right)$, where we know that

$$y_1^{(n)}(t) = -\left[p_1(t)\, y_1^{(n-1)}(t) + \cdots + p_{n-1}(t)\, y_1'(t) + p_n(t)\, y_1(t)\right]$$

and the same for $y_2^{(n)}(t), ..., y_n^{(n)}(t)$. By the **expansion property of determinant**, we have

$$\frac{dW}{dt}(t) = \det(M(t)) = -p_1(t)\, W(t), \quad \forall\, t \in I. \tag{61}$$

The proof is done. $\square$

Remark 0.38 *To understand the identity (61), we can look at the case $n = 3$ and verify it. Note that*

$$\frac{dW}{dt}(t) = \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}(t)$$
$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1y_1'' - p_2y_1' - p_3y_1 & -p_1y_2'' - p_2y_2' - p_3y_2 & -p_1y_3'' - p_2y_3' - p_3y_3 \end{vmatrix}(t).$$

We know that if we multiply the first row by $p_3(t)$ and add it onto the third row, the determinant is unchanged. Similarly, if we multiply the second row by $p_2(t)$ and add it onto the third row, the determinant is unchanged. By this, we have

$$\frac{dW}{dt}(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1y_1'' & -p_1y_2'' & -p_1y_3'' \end{vmatrix}(t) = -p_1(t)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}(t) = -p_1(t)\, W(t)$$

for all $t \in I$. Hence (61) is verified.

Theorem 0.39 *Let $y_1(t), ..., y_n(t)$ be a set of solutions to the corresponding homogeneous equation*

$$y^{(n)}(t) + p_1(t)\, y^{(n-1)}(t) + \cdots + p_{n-1}(t)\, y'(t) + p_n(t)\, y(t) = 0, \quad t \in I. \tag{62}$$

Then the family of solutions

$$y(t) = c_1y_1(t) + \cdots + c_ny_n(t), \quad t \in I \tag{63}$$

with ***arbitrary*** *real constants $c_1, ..., c_n$ includes* ***every solution*** *of (62) on I* ***if and only if***

$$W(y_1, \ldots, y_n)(t_0) \neq 0 \quad \text{for some} \quad t_0 \in I \tag{64}$$

(hence $W(y_1, \ldots, y_n)(t) \neq 0$ for all $t \in I$).

Remark 0.40 *In the above theorem, we call*

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t), \quad t \in I \tag{65}$$

the ***general solution*** *of equation (62).*

Proof. The idea of proof is similar to the previous case for second order linear equations (we need to use the **existence and uniqueness** property for equation (62) with initial conditions (57)). We omit it. □

Corollary 0.41 *Let $y_1(t), \ ..., \ y_n(t)$ be from the above theorem such that $W(y_1, \ ... \ , \ y_n)(t_0) \neq 0$ for some $t_0 \in I$. Also let $y_p(t)$ be a solution to the nonhomogeneous equation (56) on I. Then the* ***general solution*** *of equation (56) is given by*

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + y_p(t), \quad t \in I \tag{66}$$

for some constants $c_1, \ ..., \ c_n$.

Proof. We omit this. □

Section 4.2: Homogeneous equations with constant coefficients.

Remark 0.42 *Just follow textbook for this section. See p. 229 for "real and unequal roots" and p. 230-232 for "complex and repeated roots".*

In this section, we look at an n-th order linear homogeneous equation with **constant coefficients**, given by

$$L[y] := a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y^{(1)}(t) + a_n y(t) = 0, \quad t \in (-\infty, \infty), \tag{67}$$

where $a_0, \ ..., \ a_n$ are constants with $a_0 \neq 0$. similar to the previous situation, the polynomial equation

$$p_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0 \tag{68}$$

is called the **characteristic equation** of the ODE. If we plug the function $y(t) = e^{rt}$ (r is a number, which can be real or complex) into (67), we get

$$L\left[e^{rt}\right] = p_n(r) e^{rt} = 0. \tag{69}$$

Therefore, if r is a **real** root of the polynomial equation $p_n(x) = 0$, the function $y(t) = e^{rt}$ is a **real solution** of (67). If $r = \alpha + i\beta$ ($\alpha \in \mathbb{R}, \ \beta > 0$) is a **complex** root of the polynomial equation $p_n(x) = 0$, then the function $y(t) = e^{rt}$ is a **complex solution** of (67). In this case, another root must be $\bar{r} = \alpha - i\beta$ and we get another **complex solution** $y(t) = e^{\bar{r}t}$. By looking at the real and imaginary parts of the complex solutions e^{rt} and $e^{\bar{r}t}$, where $r = \alpha + i\beta$, the corresponding **two real solutions** for the two complex roots $r = \alpha + i\beta, \ \bar{r} = \alpha - i\beta$ are

$$e^{\alpha t} \cos \beta t, \qquad e^{\alpha t} \sin \beta t. \tag{70}$$

For the case of **repeated roots** (real or complex), the discussion is the same as that in Chapter 3. We omit it. It suffices to look at some examples.

For your information, there is a **decomposition property** for polynomials with **real coefficients**, which says:

Lemma 0.43 *Let $p_n(x)$ be a polynomial with degree $n \in \mathbb{N}$ given by (68). Then it has a unique (up to permutation of factors) decomposition of the form*

$$p_n(x) = a_0 (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_m)(x - z_1)(x - \bar{z}_1)\cdots(x - z_k)(x - \bar{z}_k),$$

*where $\lambda_1, \ldots, \lambda_m$ are real numbers (**may be repeated**) and $z_1, \ldots, z_k$ are complex numbers (**may be repeated**).*

If we let D denote the operator $D = d/dx$, then equation (67) can be written as

$$p_n(D)\, y = 0, \quad \text{where} \quad p_n(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

and then decompose $p_n(D)$ as

$$p_n(D) = a_0 (D - \lambda_1)(D - \lambda_2)\cdots(D - \lambda_m)(D - z_1)(D - \bar{z}_1)\cdots(D - z_k)(D - \bar{z}_k)$$

and similar to the case $n = 2$, one can decompose an n-th order linear homogeneous equation (with **constant coefficients**) into n **first order linear equations**. Thus, by induction, the solution formula for an n-th order linear homogeneous equation (with **constant coefficients**) is similar to that of a second order linear homogeneous equation (with **constant coefficients**).

Example 0.44 *(**This is for p. 233, Example 4.**) Let $p(x) = x^4 + 1$. Then we can write it as (completing the square)*

$$p(x) = x^4 + 2x^2 + 1 - 2x^2 = \left(x^2 + 1\right)^2 - \left(\sqrt{2}x\right)^2,$$

which gives the decomposition

$$p(x) = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right).$$

Hence the four roots of the equation $x^4 + 1 = 0$ are

$$\frac{-\sqrt{2} \pm \sqrt{2}i}{2}, \qquad \frac{\sqrt{2} \pm \sqrt{2}i}{2}.$$

*One can also use **complex exponential function** $e^{i\theta}$ to find the four roots of $x^4 + 1 = 0$, or use the following method: decompose $x^4 + 1 = (x^2 + i)(x^2 - i)$ and then find $a, b \in \mathbb{R}$ satisfying*

$$(a + ib)^2 = i, \qquad (a + ib)^2 = -i$$

respectively. Also see the discussion in p. 233 of the book.

Example 0.45 *Do Examples 1, 2, 3 in p. 229-233 of the book.*

Example 0.46 *Find the general solution of the equation (67) where $n = 14$ and the 14 roots of the characteristic equation $p_n(x) = 0$ are given by*

$$0,\ 0,\ -4,\ 7,\ -5,\ -5,\ -5,\ -5,\ 3+2i,\ 3+2i,\ 3+2i,\ 3-2i,\ 3-2i,\ 3-2i.$$

The answer is

$$y(t) = \begin{cases} c_1 + c_2 t + c_3 e^{-4t} + c_4 e^{7t} + \left(c_5 + c_6 t + c_7 t^2 + c_8 t^3\right) e^{-5t} \\ + \left(c_9 + c_{10} t + c_{11} t^2\right) e^{3t} \cos 2t + \left(c_{12} + c_{13} t + c_{14} t^2\right) e^{3t} \sin 2t, \end{cases}$$

where $t \in (-\infty, \infty)$..

Section 4.3: The method of undetermined coefficients.

Just follow textbook for this section. At the same time, review Table 3.5.1 in p. 182 of the book.

Example 0.47 *Do Examples 1, 2, 3, in p. 237-238, of the book.*

Remark 0.48 *Suppose we have an equation of the form*

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y^{(1)}(t) + a_n y(t) = 4t^2 \cdot \underbrace{e^{2t}\cos 8t} - 7t^5 \cdot \underbrace{e^{2t}\sin 8t},$$

then we can ***combine the two functions on the left hand side together*** *and try $y_p(t)$ to have the form*

$$y_p(t) = \begin{cases} t^s \left\{c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5\right\} \underbrace{e^{2t}\cos 8t} \\ +t^s \left\{\tilde{c}_0 + \tilde{c}_1 t + \tilde{c}_2 t^2 + \tilde{c}_3 t^3 + \tilde{c}_4 t^4 + \tilde{c}_5 t^5\right\} \underbrace{e^{2t}\cos 8t}, \end{cases}$$

*where s is the number of times (**multiplicity**) that $2+8i$ is a root of the characteristic equation.*

Section 4.4: The method of variation of parameters.

Remark 0.49 ***(Be careful.)*** *Throughout this section, we will focus on equation (71), which has leading coefficient 1 for $y'''(t)$.*

For simplicity of discussion, we only explain the method for a **third order** differential equation. **The discussion for higher order differential equation is similar.** Consider a linear differential equation given by (note that here the **leading coefficient of** $y'''(t)$ **is** 1)

$$y'''(t) + p(t) y''(t) + q(t) y'(t) + r(t) y(t) = g(t), \quad t \in I, \tag{71}$$

where $p(t),\ q(t),\ r(t),\ g(t)$ are continuous on I and $g(t)$ can be **arbitrary**. Assume that we already know a **fundamental set** of solutions $y_1(t),\ y_2(t),\ y_3(t)$ for the corresponding homogeneous equation and we want to find a **particular solution** $y_p(t)$ of (71). Similar to the second order equation, we try $y_p(t)$ to be of the form

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t) + u_3(t) y_3(t) \tag{72}$$

and compute

$$y_p'(t) = \left(\underbrace{u_1' y_1 + u_2' y_2 + u_3' y_3}\right) + \left(u_1 y_1' + u_2 y_2' + u_3 y_3'\right)$$

and we first assume that

$$\underbrace{u_1' y_1 + u_2' y_2 + u_3' y_3} = 0. \tag{73}$$

By this, we get

$$y_p''(t) = \left(\underbrace{u_1' y_1' + u_2' y_2' + u_3' y_3'}\right) + \left(u_1 y_1'' + u_2 y_2'' + u_3 y_3''\right)$$

and then we assume that

$$\underbrace{u_1' y_1' + u_2' y_2' + u_3' y_3'} = 0. \tag{74}$$

By this, we get

$$y_p'''(t) = \left(\underbrace{u_1' y_1'' + u_2' y_2'' + u_3' y_3''}\right) + \left(u_1 y_1''' + u_2 y_2''' + u_3 y_3'''\right).$$

Finally, if we assume that

$$\underbrace{u_1' y_1'' + u_2' y_2'' + u_3' y_3''} = g, \tag{75}$$

then we will get

$$y_p'''(t) + p(t)\, y_p''(t) + q(t)\, y_p'(t) + r(t)\, y_p(t)$$
$$= \left\{ \begin{array}{l} [g(t) + (u_1 y_1''' + u_2 y_2''' + u_3 y_3''')] + p(t)\,[u_1 y_1'' + u_2 y_2'' + u_3 y_3''] \\ +q(t)\,[u_1 y_1' + u_2 y_2' + u_3 y_3'] + r(t)\,[u_1 y_1 + u_2 y_2 + u_3 y_3] \end{array} \right. = g(t), \quad t \in I.$$

Thus we have found a particular solution if the above three assumptions (73), (74), (75) can be fulfilled.

In conclusion, we need to solve the system

$$\begin{cases} u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' = 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = g \end{cases} \tag{76}$$

and get (use **Cramer's rule**)

$$u_1'(t) = \frac{g(t) \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} (t)}{W(t)}, \quad u_2'(t) = \frac{g(t) \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} (t)}{W(t)}, \quad u_3'(t) = \frac{g(t) \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} (t)}{W(t)}. \tag{77}$$

A particular solution satisfying $y_p(t_0) = y_p'(t_0) = y_p''(t_0) = 0$ is given by

$$y_p(t) = \sum_{m=1}^{3} \left(\int_{t_0}^{t} \frac{g(s)\, W_m(s)}{W(s)} ds \right) y_m(t), \quad t \in I, \tag{78}$$

where

$$W_1(s) = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} (s), \quad W_2(s) = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} (s), \quad W_3(s) = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} (s). \tag{79}$$

To prove that $y_p(t)$ given by (78) does satisfy $y_p(t_0) = y_p'(t_0) = y_p''(t_0) = 0$. We observe the following:

Lemma 0.50 *We have the following identity:*

$$\sum_{m=1}^{3} W_m(t)\, y_m(t) = \sum_{m=1}^{3} W_m(t)\, y_m'(t) = 0, \quad \forall\, t \in I. \tag{80}$$

Remark 0.51 *However, we do not have* $\sum_{m=1}^{3} W_m(t)\, y_m''(t) = 0$ *for all* $t \in I$.

Proof. We first have

$$\sum_{m=1}^{3} W_m(t)\, y_m(t)$$
$$= W_1(t)\, y_1(t) + W_2(t)\, y_2(t) + W_3(t)\, y_3(t)$$
$$= \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ y_1 & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & y_2 & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & y_3 \end{vmatrix}$$
$$= y_1(y_2 y_3' - y_2' y_3) - y_2(y_1 y_3' - y_1' y_3) + y_3(y_1 y_2' - y_1' y_2) = 0, \quad \forall\, t \in I,$$

and similarly

$$\sum_{m=1}^{3} W_m(t)\, y_m'(t)$$
$$= y_1'(y_2 y_3' - y_2' y_3) - y_2'(y_1 y_3' - y_1' y_3) + y_3'(y_1 y_2' - y_1' y_2) = 0, \quad \forall\, t \in I.$$

□

Corollary 0.52 *The particular solution $y_p(t)$ given by (78) satisfies $y_p(t_0) = y_p'(t_0) = y_p''(t_0) = 0$.*

Proof. By (80) we have

$$\begin{aligned} y_p'(t) &= \sum_{m=1}^{3} \frac{g(t) W_m(t)}{W(t)} y_m(t) + \sum_{m=1}^{3} \left(\int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} ds \right) y_m'(t) \\ &= \sum_{m=1}^{3} \left(\int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} ds \right) y_m'(t), \quad \forall\, t \in I, \end{aligned} \tag{81}$$

and also

$$\begin{aligned} y_p''(t) &= \sum_{m=1}^{3} \frac{g(t) W_m(t)}{W(t)} y_m'(t) + \sum_{m=1}^{3} \left(\int_{t_0}^{t} \frac{g(t) W_m(t)}{W(t)} ds \right) y_m''(t) \\ &= \sum_{m=1}^{3} \left(\int_{t_0}^{t} \frac{g(t) W_m(t)}{W(t)} ds \right) y_m''(t), \quad \forall\, t \in I. \end{aligned} \tag{82}$$

The above two identities imply $y_p(t_0) = y_p'(t_0) = y_p''(t_0) = 0$. □

We conclude the following:

Theorem 0.53 *Consider the third order linear nonhomogeneous equation*

$$y'''(t) + p(t)\, y''(t) + q(t)\, y'(t) + r(t)\, y(t) = g(t), \quad t \in I, \tag{83}$$

where $p(t),\ q(t),\ r(t),\ g(t)$ are constant function on I with initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \quad y''(t_0) = \gamma_0.$$

Then the unique solution is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) + y_p(t), \quad t \in I,$$

where the above $y_p(t)$ is from (78) and the constants $c_1,\ c_2,\ c_3$ satisfy the equations

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) + c_3 y_3(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + c_3 y_3'(t_0) = z_0 \\ c_1 y_1''(t_0) + c_2 y_2''(t_0) + c_3 y_3''(t_0) = \gamma_0. \end{cases} \tag{84}$$

Remark 0.54 *In case there is no initial conditions, we can use the* ***indefinite integral formula*** *for $y_p(t)$:*

$$y_p(t) = \sum_{m=1}^{3} \left(\int \frac{g(t) W_m(t)}{W(t)} dt \right) y_m(t), \quad t \in I \tag{85}$$

and obtain the general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) + y_p(t), \quad t \in I.$$

Example 0.55 *Do Example 1 of the book in p. 243.*

Example 0.56 *(**This is Problem 1 in p. 244**.) Solve the equation $y''' + y' = 2\tan t$, where $t \in (-\pi/2, \pi/2)$.*

<u>**Solution**</u>:

The solutions for the homogeneous equation are $y_1(t) = 1$, $y_2(t) = \cos t$, $y_3(t) = \sin t$, and their Wronskian is given by

$$W(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1$$

and we also have

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1, \quad W_2(t) = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t$$

and

$$W_3(t) = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t.$$

Therefore, we conclude (we use the indefinite integral formula)

$$\begin{aligned} y_p(t) &= \sum_{m=1}^{3} \left(\int \frac{g(t) W_m(t)}{W(t)} dt \right) y_m(t) = \sum_{m=1}^{3} \left(\int (2\tan t) W_m(t)\, dt \right) y_m(t) \\ &= \int (2\tan t)\, dt + \left(\int (2\tan t)(-\cos t)\, dt \right) \cos t + \left(\int (2\tan t)(-\sin t)\, dt \right) \sin t \\ &= -2\log(\cos t) + 2\cos^2 t + \left(-2 \int \frac{1}{\cos t} dt + 2\sin t \right) \sin t \\ &= -2\log(\cos t) + 2\cos^2 t - (2\log|\sec t + \tan t|)\sin t + 2\sin^2 t \\ &= -2\log(\cos t) - (2\log|\sec t + \tan t|)\sin t + 2, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right). \end{aligned}$$

Thus the **general solution** of the equation $y''' + y' = 2\tan t$ over the interval $(-\pi/2, \pi/2)$ is given by (we absorb the constant 2 into c_1)

$$y(t) = c_1 + c_2 \cos t + c_3 \sin t - 2\log(\cos t) - (2\log|\sec t + \tan t|)\sin t, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right),$$

where c_1, c_2, c_3 are arbitrary constants. □